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Robust structured H_2 synthesis for linear systems subject to time-invariant uncertainties with global optimization

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AbstractThis paper proposes a new way to solve the structured H_2 synthesis problem under time invariant plant uncertainties. A worst-case minimization approach enables to formulate the synthesis problem as a min-max problem subject to quantified constraints. We propose a global optimization algorithm based on Interval Analysis to solve this problem. This algorithm provides an upper and a lower bound on a worst-case H_2 norm objective function. The originality of the paper relies on the reformulation of the optimization problem as a min-max problem, providing bounds on the worst-case H_2 norm.

Keywords: Robust control, Global Optimisation, H_2-norm

1. INTRODUCTION

Optimization has played a central role in control system theory for a very long time. During the last decades, a large panel of control problem has been formulated into optimization problem and particularly in convex optimization based on Linear Matrix Inequality (LMI) problems, see Nesterov and Nemirovskii (1994); Boyd et al. (1994); Ghaoui and Niculescu (2000).

However, synthesis problem formulation generally leads to a non-convex optimization problem, especially when robustness against uncertainties must be taken into account. When considering uncertain plants, LMI solutions were given for the control synthesis problems but generally induce conservatism or relaxation, see Scherer and Kose (2006); Farges et al. (2007); Karimi et al. (2008). On the other hand, several approaches were proposed to solve the non-convex synthesis problems directly with uncertainties. Indeed, in Monnet et al. (2016a), performance criteria such as H_∞ norm or stability can be expressed as polynomial inequalities, see Dorato (2000). Therefore, set-membership methods such as symbolic computation or interval analysis can be used to deal with the uncertainties, see Anai and Hara (2002); Malan et al. (1997).

In the present work, we propose to solve the robust H_2 synthesis problem, addressed as a worst-case minimization problem. A similar way has been developed for the H_∞ synthesis, see Monnet et al. (2016a, 2017). We minimize, over a set of controllers, the maximum over a set of uncertainties a H_2 criterion, and the problem is formulated as follows:

$$\min_{K} \sup_{\theta\in\Theta}(||G(\theta)||_2) \text{ s.t. } K \text{ ensures internal stability.} \quad (1)$$

where Θ represents the set of uncertainties and G the feedback system closed with K the controller. In a general case if the controller is structured, Problem (1) is non-convex. As a consequence, we consider a global optimization approach to tackle the problem, which provides a guaranteed enclosure of the minimum, see Kearfott (1992). Addressing the H_2 synthesis with global optimization contrast with the other existing approach based on non-smooth optimization [Apkarian et al. (2014); Ravanbod-Hosseini et al. (2011)], convex optimization [Boyd et al. (1994)], or genetic algorithm [Bianco and Piazzi (1998)]. As far as the authors know, the robust H_2 synthesis has never been addressed by global optimization methods.

This paper is organized as follows. Section 2 first gives the formulation of the robust H_2 synthesis problem as a min-max problem and recalls the specific H_2-norm formula which is used. In Section 3, we then introduced a global optimization algorithm to solve min-max problems; the algorithm is described and we discuss the equivalence with the H_2 synthesis problem. In order to illustrate the behavior of the proposed method, Section 4 deals with a numerical example of literature of Bianco and Piazzi (1998), and compared with other H_2 synthesis methods. This example can be applied to robust optimal control of an AUV as proposed in Qiao et al. (2018); Moreira and Guedes Soares (2008). This work is a new step in the use of global optimization in robust control and the conclusion gives some future works.

2. PROBLEM STATEMENT: STRUCTURED H_2 SYNTHESIS UNDER PARAMETRIC UNCERTAINTIES

In this section, we recall the H_2-norm expression of a Linear Time Invariant (LTI) system. Then, we formulate

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the H_2 synthesis problem under parametric uncertainties as a non-convex min-max optimization problem.

2.1 H_2 norm computation

We consider a stable and strictly proper Multiple Input Multiple Output (MIMO) LTI system $G(s)$ with the following expression:

$$G(s) = [G_{ij}(s)]_{\substack{1\le i\le p\\1\le j\le q}}$$

where $G_{ij}(s)$ denotes the transfer function from u_i to y_j.

Proposition 1. (Popov, 1962, p. 367) Let $y(t)$ be the impulse response of a SISO system $G(s)$,

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + ... + b_m}{a_0 s^n + a_1 s^{n-1} + ... + a_n} = \frac{N(s)}{D(s)}.$$

The H_2 norm of $G(s)$ is equal to the $\mathcal{L}_2$ norm of $y(t)$ given by

$$||G(s)||_2^2 = ||y||_2^2 = \int_0^\infty y^2(t)dt = \frac{(-1)^m \det(C)}{a_0\Delta}, \quad (2)$$

where Δ is the Hurwitz determinant of $D(s)$ and

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & ... & c_{n-1} \\ a_0 & a_2 & a_4 & ... & 0 \\ 0 & a_1 & a_3 & ... & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & ... & a_n \end{pmatrix},$$

with $N(s)N(-s) = c_0 s^{2m} + c_1 s^{2(m-1)} + ... + c_{m-1}s^2 + c_m$.

Using Proposition 1, the square of H_2 norm of each G_{ij} can be expressed as a rational function. Moreover, we have the classical expression given by Equation (3).

$$||G(s)||_2 = \sqrt{\sum_{ij} ||G_{ij}(s)||_2^2}. \quad (3)$$

As a consequence, the H_2 norm of $G(s)$ is equal to the square root of a rational function. This remark is the starting point for using our global optimization algorithm.

2.2 Structured H_2 synthesis under parametric uncertainties

Consider a structured controller $K(k, s)$ which depends on tunable variables $k \in \mathbb{R}^n$, and a LTI system $G(s)$. A structured controller, for example a Proportional Integral (PI), is generally preferred over unstructured controller provided by LMI solution Banjerdpongchai and How (1996); Stoorvogel (1993); Nagamune et al. (2006), since the user can explicitly choose the structure of the controller. Let the closed-loop system represented in Figure 1 be described by the Linear Fractional Transform (LFT) of G with K, $F(G, K)$. The vector of exogenous inputs is denoted w and the vector of outputs z. The structured H_2 synthesis consists in minimizing the H_2 norm of $F(G, K(k))$, i.e. solving the optimization Problem (4).

$$\min_{k\in\mathbb{R}^n} ||F(G, K(k))||_2 \quad (4)$$

Suppose now that $G(\theta, s)$ depends on time invariant uncertain parameters $\theta \in \Theta$, where $\Theta \subset \mathbb{R}^n$ is a bounded set. One way to deal with the uncertainties is to minimize the worst-case over Θ of $||F(G(\theta), K(k))||_2$. The structured H_2 synthesis problems under uncertainties can be formulated as the minmax Problem (5).

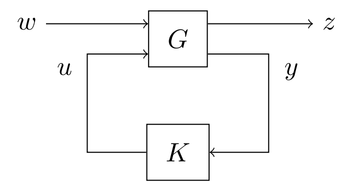

Figure 1. Standard regulation scheme.

$$\min_{k\in\mathbb{R}^n} \left(\sup_{\theta\in\Theta} ||F(G(\theta), K(k))||_2 \right) \quad (5)$$

Note that Problem (5) is non-convex in the general case. In addition, we also impose the closed loop system to be robustly stable. The robust stability can be expressed, with the Routh-Hurwitz criterion, as a set of Semi-Infinite Constraints, involving polynomial inequalities which depend on θ and k. Thus, Problem (5) becomes a quantified-constrained min-max problem, given by Problem (6).

$$\begin{cases} \min\limits_{k\in\mathbb{R}^n} & \sup\limits_{\theta\in\Theta} ||F(G(\theta), K(k))||_2 \\ \text{subject to} & \forall\theta \in \Theta,\ R(k,\theta) \le 0 \end{cases} \quad (6)$$

where $R(k, \theta)$ denotes the polynomials of the Routh-Hurwitz criterion.
We propose to tackle Problem (6) with a global optimization approach presented in the next section.

3. GLOBAL OPTIMIZATION ALGORITHM

This section proposes a global optimization algorithm to solve Problem (6). The algorithm is based on an Interval Branch and Bound Algorithm (IBBA), and provides an upper bound and a lower bound on the minimum of the worst-case H_2 norm of the closed loop system, see Kearfott (1992). For the sake of a better understanding of the algorithm, we propose to first introduce Interval Arithmetic and then to show how it is used for a branch and bound approach.

3.1 Interval Analysis

We first define intervals and introduce interval arithmetic, for more details see Jaulin et al. (2001).

Definition 1. An interval is a closed connected subset of $\mathbb{R}$ Moore et al. (2009), described by its endpoints $\underline{x}$ and $\overline{x}$:

$$\boldsymbol{x} = [\underline{x}, \overline{x}] = \{x \mid \underline{x} \le x \le \overline{x}\},$$
$$\text{with } \underline{x} \in \mathbb{R}\cup\{-\infty\} \text{ and } \overline{x} \in \mathbb{R}\cup\{+\infty\}.$$

The set of intervals is denoted by $\mathbb{IR}$ and the set of n-dimensional interval vectors, also called boxes, is denoted by $\mathbb{IR}^n$. In this paper, the intervals are noted in bold text. All the common operators such as $+, -, \times, /, \sqrt{}, ...$ are defined on $\mathbb{IR}$. For example:

$$[1, 2] + [-1, 2] = [0, 4]$$
$$[1, 2] - [-1, 2] = [-1, 3]$$
$$[1, 2] \times [-1, 2] = [-2, 4]$$
$$[2, 4]/[0, 2] = [1, +\infty]$$

Definition 2. Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a function and $\boldsymbol{x} \in \mathbb{IR}^n$. An inclusion function $\boldsymbol{f}$ of f is defined from $\mathbb{IR}^n$ into $\mathbb{IR}^m$ and respects the following property:

$$f(\boldsymbol{x}) = \{f(x),\ x \in \boldsymbol{x}\} \subseteq \boldsymbol{f}(\boldsymbol{x}).$$

Remark 1. $\boldsymbol{f}(\boldsymbol{x})$ may not be the smallest box that encloses $f(\boldsymbol{x})$. The inclusion function generally provides a guaranteed outer approximation.

Given a function f of several variables $x_1, \ldots, x_n$ and the corresponding intervals for the variables $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, the *natural interval extension* $\boldsymbol{f}$ of f is an interval obtained by substituting variables by their corresponding intervals and applying the interval arithmetic operations. The natural interval extension of a function provides an *inclusion function*.

As a consequence, it is possible to provide guaranteed bounds of a function over an interval.

3.2 Interval Branch and Bound Algorithm

An IBBA is a global optimization algorithm based on inclusion functions, see Ninin et al. (2015). Consider a constrained minimization problem formulated by Problem (7):

$$\begin{cases} \min\limits_{x \in \boldsymbol{x}} & f(x), \\ \text{subject to} & g(x) \leq 0, \end{cases} \tag{7}$$

where f and g are continuous functions from $\mathbb{R}^n$ into $\mathbb{R}$, and $\boldsymbol{x} \subseteq \mathbb{IR}^n$.

The IBBA provides the best found solution $\tilde{x}$ and a guaranteed enclosure $\boldsymbol{\mu}$ of the global minimum value of Problem (7).
Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be inclusion functions of f and g respectively. Let $\mathcal{L}$ be a list of boxes, $\epsilon > 0$, lb_μ and ub_μ be three scalars. The IBBA is described by Algorithm 1, and works as follows. At the initialization, $\mathcal{L}$ contains the box $\boldsymbol{x}$.

Algorithm 1. Branch and Bound algorithm.

1: $\mathcal{L} = \{\boldsymbol{x}\}$, $ub_\mu = +\infty$, $lb_\mu = -\infty$.
2: **while** $ub_\mu - lb_\mu \geq \epsilon$ and $\mathcal{L} \neq \emptyset$ **do**
3: Extract a box $\boldsymbol{x}$ from $\mathcal{L}$.
4: Bissect $\boldsymbol{x}$ into $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$.
5: **for** $i = 1, 2$ **do**
6: **if** $\underline{\boldsymbol{g}(\boldsymbol{x}_i)} \leq 0$ **then**
7: Compute $\boldsymbol{f}(\boldsymbol{x}_i)$.
8: **if** $\underline{\boldsymbol{f}(\boldsymbol{x}_i)} \leq ub_\mu$ **then**
9: Add $\boldsymbol{x}_i$ in $\mathcal{L}$.
10: **end if**
11: Choose $x' \in \boldsymbol{x}_i$ that satisfy $g(x') \leq 0$.
12: **if** $f(x') < ub_\mu$ **then**
13: $\tilde{x} = x'$.
14: $ub_\mu = f(\tilde{x})$.
15: **end if**
16: **end if**
17: **end for**
18: **end while**
19: Update $lb_\mu = \min\limits_{\boldsymbol{x} \in \mathcal{L}} \underline{\boldsymbol{f}(\boldsymbol{x})}$.

A box $\boldsymbol{x}$ is extracted from the list $\mathcal{L}$ and divided into two non-overlapping sub-boxes $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, such as $\boldsymbol{x} = \boldsymbol{x_1} \cup \boldsymbol{x_2}$ and $\boldsymbol{x_1} \cap \boldsymbol{x_2} = \emptyset$. Then, the algorithm processes $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ the same way. At line 6, we check if $\boldsymbol{x}_i$ does not satisfy the constraint. Indeed, if $\underline{\boldsymbol{g}(\boldsymbol{x}_i)} > 0$, the constraint is proved not to be satisfied for all $x \in \boldsymbol{x}_i$ and the algorithm process the next box. At line 8, we check if $\boldsymbol{x}_i$ may contain the solution to Problem (7). Indeed, if $\underline{\boldsymbol{f}(\boldsymbol{x}_i)} > ub_\mu$, it proves that $\boldsymbol{x}_i$ cannot contain the global minimum of Problem (7) since a better solution $\tilde{x}$ was found: $\forall x \in \boldsymbol{x}_i, f(x) > f(\tilde{x})$, otherwise $\boldsymbol{x}_i$ is added to $\mathcal{L}$. At line 11, we choose a point $x' \in \boldsymbol{x}_i$ such that the constraint is satisfied. If $f(x')$ is lower than the best current minimum ub_μ, x' is the new best-known solution and $\tilde{x}$ and ub_μ are updated. When Algorithm 1 stops, $\mathcal{L}$ contains the global solutions of Problem (7). Therefore, the lowest lower bound of $\boldsymbol{f}$ over the boxes contained in $\mathcal{L}$ is a lower bound of the global minimum value, and lb_μ is updated at line 19. When Algorithm 1 stops, $\boldsymbol{\mu} = [lb_\mu, ub_\mu]$ is an enclosure of the global minimum value and $\tilde{x}$ is the best feasible point found. If $\mathcal{L}$ is empty, it proves that Problem (7) is not feasible.

Remark 2. If Problem (7) is subject to several constraints $g_j(x) \leq 0$, line 6 becomes:

$$\textbf{if } \forall j, \underline{\boldsymbol{g}_j(\boldsymbol{x}_i)} \leq 0.$$

If at least one constraint is not satisfied, Algorithm 1 process the next box since it is proved that no point of $\boldsymbol{x}_i$ satisfy the constraints.

Moreover, to improve the convergence time of the IBBA, we use advanced constraint propagation and global optimization techniques, for details see Ninin et al. (2015).

3.3 Resolution of robust H_2 min-max problem

Algorithm 1 provides a general understanding of global optimization based on interval analysis, and we will show how it can be used to solve Problem (6) in a first instance. Nonetheless, in Monnet et al. (2016b), a more efficient version of Algorithm 1 dedicated to min-max problems, briefly presented in this section, is used in Section 4.
In order to solve Problem (6) with Algorithm 1, we reformulate it as Problem (8) with $\mathbb{K} \subseteq \mathbb{IR}^n$ the initial set of controller parameters.

$$\begin{cases} \min\limits_{k \in \mathbb{K}} & f(k), \\ \text{subject to} & R(k, \theta) \leq 0, \, \forall \theta \in \Theta. \end{cases} \tag{8}$$

The objective function f of Problem (8) is given by Equation 9.

$$\begin{aligned} f(k) &= \sup_{\theta \in \Theta} H(k, \theta), \\ H(k, \theta) &= ||F(G(\theta), K(k))||_2. \end{aligned} \tag{9}$$

Proposition 2. The natural interval extensions of R and H provide an inclusion of R and an inclusion function of f over $\mathbb{K} \times \Theta$, respectively.

Proof. Since R is a polynomial depending on k and θ (Section 2.2), an inclusion function $\boldsymbol{R}$ of R can be defined with interval arithmetic (Section 3.1).
H is the square root of a rational function (Section 2.1), therefore an inclusion function $\boldsymbol{H}$ of H can be defined using interval analysis. Since $\boldsymbol{H}$ is an inclusion function of H, we have

$$H(k, \theta) \in \mathbf{H}(\mathbb{K}, \Theta), \, \forall (k, \theta) \in \mathbb{K} \times \Theta, \tag{10}$$

and from Equation 9, by continuity of H over Θ,

$$\exists \theta' \in \Theta \mid f(k) = H(k, \theta') = \sup_{\theta \in \Theta} H(k, \theta). \tag{11}$$

Thus, from Equation (10) and Equation (11), it follows that

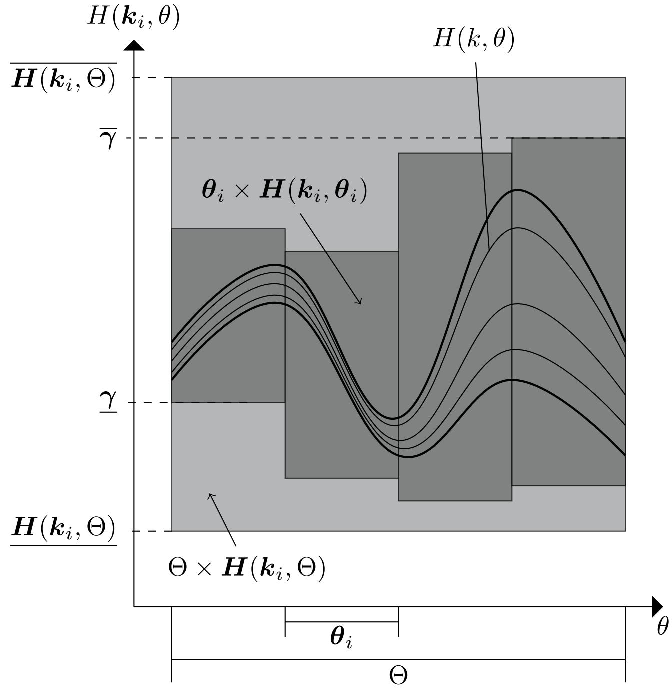

Figure 2. Computation of $f(\boldsymbol{k}_i)$.

$$\forall (k, \theta) \in \mathbb{K} \times \Theta,\ f(k) \in \mathbf{H}(\mathbb{K}, \Theta). \tag{12}$$

Equation (12) proves that $\boldsymbol{H}$ is an inclusion function of f. $\square$

Since $\boldsymbol{H}$ and $\boldsymbol{R}$ are inclusion function of f and R respectively, Algorithm 1 can be used to solve Problem (6). Indeed, the condition of line 6 becomes $\underline{\mathbf{R}(\boldsymbol{k}_i, \Theta)} \leq 0$, and at line 7, $f(\boldsymbol{k}_i)$ can be approximated by $\overline{\mathbf{H}(\boldsymbol{k}_i, \Theta)}$ as shown in Figure 2. At line 11, $k' \in \boldsymbol{k}_i$ is chosen such as $\overline{\boldsymbol{R}(k', \Theta)} < 0$, which ensure that the closed-loop system is stable for all $\theta \in \Theta$, and at line 12, Algorithm 1 check if $\overline{\boldsymbol{H}(k', \Theta)}$ is lower than ub_μ. In Figure 2, the three thin curves represent $H(k, \theta)$ for three particular elements $k \in \boldsymbol{k}_i$. The two thick curves represent the hull of the family of function $H(\boldsymbol{k}_i, \theta)$. One can remark that $\mathbf{H}(\mathbf{k_i}, \boldsymbol{\Theta})$, represented by the light gray box, provides a poor approximation of $f(\boldsymbol{k}_i)$. As a consequence, Algorithm 1 will converge to a too large enclosure of the global minimum value, and may be slow to converge.

In Monnet et al. (2016b), Algorithm 2, derived from Algorithm 1, is proposed to solve min-max problems in a more efficient way. This second IBBA is used to provide a thin enclosure $\boldsymbol{\gamma}$ of $H(\boldsymbol{k}_i, \Theta)$ instead of using bounds provided by $\boldsymbol{H}$, as shown in Figure 2. The interval Θ is bisected in smaller intervals $\boldsymbol{\theta}_i$ to provide a thin enclosure of $H(\boldsymbol{k}_i, \theta)$ over Θ, represented by the dark gray boxes. The algorithm, presented in Monnet et al. (2016b), was successfully applied to H_∞ synthesis in Monnet et al. (2016a) and robust H_∞ synthesis in Monnet et al. (2017). A similar algorithm was presented in Carrizosa and Messine (2021).

Remark 3. An IBBA based on interval analysis, such as Algorithm 1, imposes that the set of uncertainties Θ is a box. A box is, in most cases, a suited representation of parametric uncertainties.

Algorithm 2. Bounds of $H(k, \theta)$ over $\boldsymbol{k_i} \times \Theta$: $\boldsymbol{\gamma} = [\underline{\gamma}, \overline{\gamma}]$

```
ℒ = {Θ}, γ̄ = +∞, γ̲ = −∞.
while γ̄ − γ̲ ≥ ε and ℒ ≠ ∅ and nb_iter < nb_max do
    Extract a box θ from ℒ.
    Bissect θ into θ_1 and θ_2.
    for j = 1, 2 do
        Compute H(k_i, θ_j).
        if overline(H(k_i, θ_j)) ≥ γ̲ then
            Add θ_j in ℒ.
        end if
        if underline(H(k_i, θ_j)) > γ̲ then
            γ̲ = underline(H(k_i, θ_j)).
        end if
    end for
    nb_iter = nb_iter + 1.
end while
Update γ̄ = max_{θ_j ∈ ℒ} overline(H(k_i, θ_j)).
```

4. EXAMPLE

In this section, we illustrate our approach with the example proposed in Bianco and Piazzi (1998) dedicated to mixed H_∞/H_2 synthesis. We limit the example to this academic example but as shown in Qiao et al. (2018); Moreira and Guedes Soares (2008), it can be extended to the underwater vehicles. This point will be investigated later. However, we consider only the H_2 synthesis problem of this example in this section. Consider the uncertain plant

$$P(\theta, s) = \theta_1 \frac{(1 - \theta_2 s)\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

with $\delta = 0.5$, $\omega_n = 5$ and $\theta = (\theta_1, \theta_2)^T \in \Theta = [0.9, 1.1] \times [0.007, 0.01]$. The regulation scheme proposed is shown in Figure 3. The problem is to find a controller

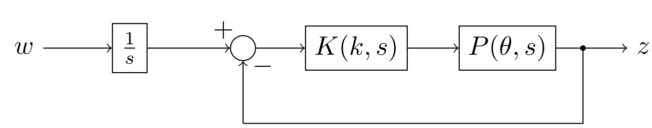

Figure 3. Proposed regulation scheme.

$$K(k, s) = \frac{k_1 s^2 + k_2 s + k_3}{s(s + 50)}$$

that minimizes the worst-case H_2 norm of the transfer from w to z and robustly stabilizes the closed-loop system. We propose to search the tunable parameters $k = (k_1, k_2, k_3)^T$ in the set $\mathbb{K} = [0, 50] \times [0, 50] \times [0, 50]$. Thus, the problem can be formulated as Problem (13),

$$\begin{cases} \min\limits_{k \in \mathbb{K}} & \left(\sup\limits_{\theta \in \Theta} ||T_{w \to z}(k, \theta)||_2 \right), \\ \text{subject to} & R(k, \theta) \leq 0\,,\, \forall \theta \in \Theta, \end{cases} \tag{13}$$

where $T_{w \to z}$ denotes the transfer from w to z.
With our algorithm, we obtain:

$$||T_{w \to z}(k^*, \theta^*)||_2 \in [0.561, 0.569],$$

where k^* denotes the global solution of Problem (13), and θ^* the worst-case over Θ at k^*.
The best solution found is $\tilde{k} = (47.598, 49.885, 49.885)^T$ which gives $||T_{w \to z}(\tilde{k}, \tilde{\theta}^*)||_2 = 0.569$. Figure 4 represents the Bode diagram of $T_{w \to z}(\tilde{k}, \theta)$ for five random values

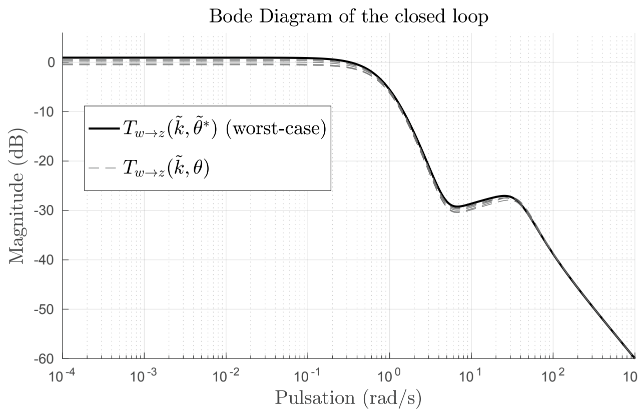

Figure 4. Magnitude of the objective channel.

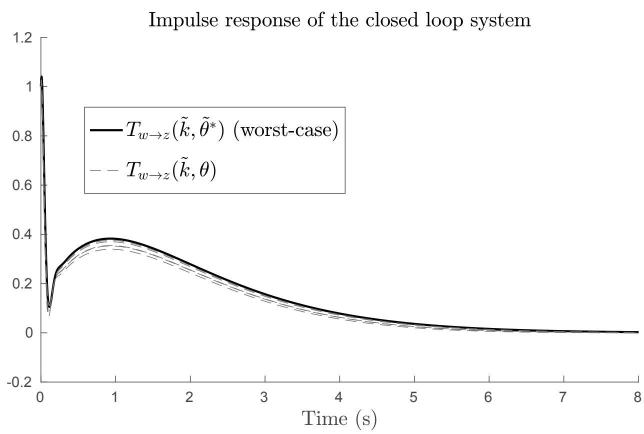

Figure 5. Time response of the closed loop to an impulse.

of θ by dashed lines, and also the worst-case obtain at $\tilde{\theta}^* = (0.9, 0.01)^T$ by the solid line. One can remark on Figure 4 that the transfer of the worst-case is greater than the transfers displayed for random values of uncertainty from 10^{-4} rad/s to 50 rad/s. For pulsations greater than 50 rad/s, the worst case transfer is not greater than all the others. Recall that the H_2 norm is an integral over the pulsation of the modulus of a transfer function

$$||T_{w \to z}(k, \theta)||_2 = \sqrt{\frac{1}{2\pi} \int_{+\infty}^{-\infty} |T_{w \to z}(k, \theta, i\omega)| \mathrm{d}\omega},$$

it is not evident to infer that $T_{w \to z}(\tilde{k}, \tilde{\theta}^*)$ is the worst-case from the frequency plot. That is why the time response of the closed loop system to an impulse is displayed on Figure 5. Figure 5 displays the time responses of the closed loop systems to a Dirac signal.

In order to verify the robust stability of the closed-loop system, the poles of $T_{w \to z}(\tilde{k}, \theta)$ are displayed on Figure 6 for thirty values of uncertainty θ. The two conjugate poles lies in the left half plan, which image the robust stability of the system.

We compare our result with the structured H_2 synthesis implemented in the `Matlab Systune` toolbox Apkarian et al. (2014). Since `systune` is based on local optimiza-

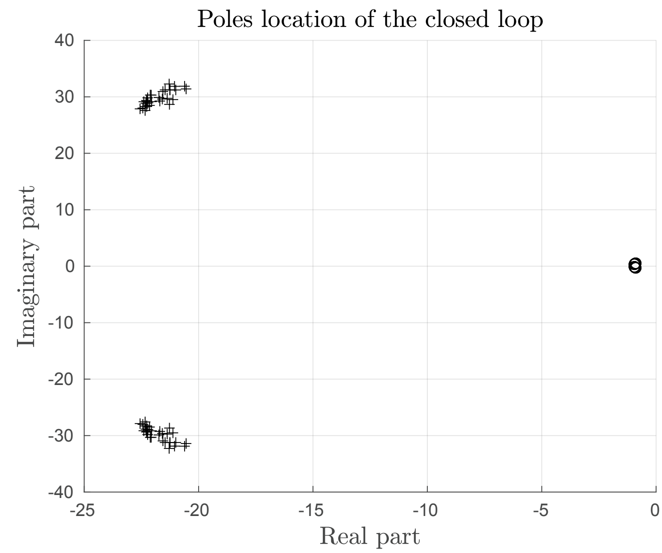

Figure 6. Poles of the closed loop system.

Method	$\sup_{\theta \in \Theta} \|\|T_{w \to z}\|\|_2$	k	CPU (s)
Global Optim.	$[0.56, 0.57]$	$\begin{pmatrix} 47.6 \\ 49.9 \\ 49.9 \end{pmatrix}$	98
`Systune`	0.567	$\begin{pmatrix} 50 \\ 50 \\ 50 \end{pmatrix}$	93

Table 1. Synthesis method comparison.

tion, the optimization process is run several times so that the total computation time correspond to the computation time of the global optimization algorithm. `systune` provides the structured controller $K(k_{st}, s)$ with $k_{st} = (50, 50, 50)^T$ for the nominal plant value $T_{w \to z}(k, \theta_n)$, with $\theta_n = (0.0085, 1)$. The H_2 norm of the nominal closed loop transfer achieve at (k_{st}, θ_n) is equal to 0.527, that is $||T_{w \to z}(k_{st}, \theta_n)||_2 = 0.527$. With our global optimization approach, we can compute the worst-case over Θ of the H_2 norm of $T_{w \to z}(k_{st}, \theta)$ by solving the "sup" part of Problem (13) (see Section 3.3), and verify if $R(k_{st}, \theta) \leq 0$, $\forall \theta \in \Theta$. We obtain $\boldsymbol{R}(k_{st}, \Theta) = [-0.99, -0.99]$, which implies that k_{st} satisfies the robust stability constraint of Problem (13), and $||T_{w \to z}(k_{st}, \theta^*_{st})||_2 = 0.567$, where θ^*_{st} the worst case value of θ over Θ of the H_2 norm at k_{st}. Table 1 summarizes the results obtained with the two synthesis methods.

One can remark that the controller provided by `Systune` is slightly better than the one computed with our Global Optimization algorithm, and is consistent with the enclosure of the global minimum computed by our method. The lower bound of this enclosure is of major importance since it proves that any controller ensuring robust stability can achieve a worst-case H_2 norm lower than 0.56. As a consequence, both $\tilde{k}$ and k_{st} provide values of H_2 norm close to the optimum value.

The value of the worst-case H_2 norm at k_{sys} is obtained with the global optimization tools introduced in this paper, which highlights the fact that global optimization can be used for robustness analysis as well as for robust synthesis.

5. CONCLUSIONS AND FUTURE WORKS

5.1 Conclusions

In this paper, we propose a worst-case formulation of the robust H_2 synthesis. The problem is formulated as a constrained min-max optimization problem. The analytic expression of the H_2 norm of a LTI plant allows to use set-membership methods and to solve this non-convex problem in a global way with a branch-and-bound algorithm. The algorithm provides an enclosure of the global minimum value of worst-case H_2 norm. In addition, the algorithm also deals with quantify constraints that ensure the robust stability of the closed loop system.

Our approach proposes to synthesize a parametrized controller, leading to a non-convex synthesis problem in the general case. Since our algorithm is able to solve non-convex programs, we can provide non-conservative results, contrary to LMI based formulations. However, the complexity of a branch and bound algorithm grows exponentially in the number of the optimization variables. As a consequence, our approach is appropriate to design controller of reasonable order (PID for example) to control a plant that does not contain too many uncertain parameters (up to 3 uncertain parameters). On the contrary, LMI formulations are suited for large scale systems. Therefore, the choice of the formulation of the problem, and by extension the method use to solve it, is a trade-off between complexity and conservatism.

5.2 Future Works

In Monnet et al. (2017), a min-max formulation of the robust H_∞ synthesis problem was proposed to compute a structured controller. A simpler global optimization algorithm was used to solve the problem, since this problem does not contain quantified constraints. As a consequence, the global method introduced in this paper could be use for the robust H_2/H_∞ synthesis of structured controller.

Although our synthesis methods provide interesting results on an academic example, the algorithm must be improved to solve real case problem as proposed in [Ninin et al. (2021)]. Indeed, the application scope of robust synthesis is very wide, from autonomous underwater vehicles (AUV) [Yang et al. (2015); Maalouf et al. (2015)] and unmanned surface vehicles [Clement (2013)] to space launchers [Arzelier et al. (2006); Clement et al. (2005)].

REFERENCES

Anai, H. and Hara, S. (2002). A parameter space approach for fixed-order robust controller synthesis by symbolic computation. In *Proc. of 15th IFAC World Congress*.

Apkarian, P., Gahinet, P., and Buhr, C. (2014). Multi-model, multi-objective tuning of fixed-structure controllers. In *Proc. of European Control Conference*.

Arzelier, D., Clement, B., and Peaucelle, D. (2006). Multi-objective H_2/H_∞ Impulse-to-Peak control of a space launch vehicle. *European Journal of Control*.

Banjerdpongchai, D. and How, J. (1996). Parametric robust H_2 control design using lmi synthesis. In *Proc. of AIAA Guidance, Navigation, and Control Conference*.

Bianco, C.G.L. and Piazzi, A. (1998). A worst-case approach to siso mixed H_2/H_∞ control. In *Proc. of the IEEE International Conference on Control Applications*.

Boyd, S., Ghaoui, L.E., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*. Studies in Applied Mathematics. SIAM.

Carrizosa, E. and Messine, F. (2021). An interval branch and bound method for global robust optimization. *Journal of Global Optimization*.

Clement, B. (2013). Control algorithms for a sailboat robot with a sea experiment. In *Proc. of IFAC Conference on Control Applications in Marine Systems*.

Clement, B., Duc, G., and Mauffrey, S. (2005). Aerospace launch vehicle control: a gain scheduling approach. *Control Engineering Practice*.

Dorato, P. (2000). Quantified multivariate polynomial inequalities. the mathematics of practical control design problems. *IEEE Control Systems Magazine*.

Farges, C., Peaucelle, D., Arzelier, D., and Daafouz, J. (2007). Robust H_2 performance analysis and synthesis of linear polytopic discrete-time periodic systems via lmis. *Systems & Control Letters*.

Ghaoui, L.E. and Niculescu, S. (eds.) (2000). *Advances in linear matrix inequality methods in control: advances in design and control*. SIAM.

Jaulin, L., Kieffer, M., Didrit, O., and Walter, E. (2001). *Applied Interval Analysis*. Springer London.

Karimi, A., Galdos, G., and Longchamp, R. (2008). Robust fixed-order H_∞ controller design for spectral models by convex optimization. In *Proc. of 47th IEEE Conference on Decision and Control*.

Kearfott, R.B. (1992). An interval branch and bound algorithm for bound constrained optimization problems. *Journal of Global Optimization*.

Maalouf, D., Chemori, A., and Creuze, V. (2015). L1 adaptive depth and pitch control of an underwater vehicle with real-time experiments. *Ocean Engineering*, 98, 66–77.

Malan, S., Milanese, M., and Taragna, M. (1997). Robust analysis and design of control systems using interval arithmetic. *Automatica*, 1363–1372.

Monnet, D., Ninin, J., and Clement, B. (2016a). A global optimization approach to structured regulation design under hinfinity constraints. In *Proc. of 55th IEEE Conference on Decision and Control*.

Monnet, D., Ninin, J., and Clement, B. (2016b). Global optimization of continuous minimax problem. In *Proc. of the 13rd Global Optimization Workshop*.

Monnet, D., Ninin, J., and Clement, B. (2017). A global optimization approach to H_∞ synthesis with parametric uncertainties applied to auv control. In *Proc. of the 20th IFAC World Congress*.

Moore, R., Kearfott, R., and Cloud, M. (2009). *Introduction to interval analysis*. Society for Industrial and Applied Mathematics, Philadelphia, PA.

Moreira, L. and Guedes Soares, C. (2008). h_2 and h_∞ designs for diving and course control of an autonomous underwater vehicle in presence of waves. *IEEE Journal of Oceanic Engineering*.

Nagamune, R., Huang, X., and Horowitz, R. (2006). Robust H_2 synthesis for dual-stage multi-sensing track-following servo systems in hdds. In *Proc. of American*

Control Conference, 1284–1289.

Nesterov, Y. and Nemirovskii, A. (1994). *Interior-point polynomial algorithms in convex programming*. Siam.

Ninin, J., Messine, F., and Hansen, P. (2015). A reliable affine relaxation method for global optimization. *4OR: A Quarterly Journal of Operations Research*.

Ninin, J., Monnet, D., and Clement, B. (2021). Nested branch-and-bound algorithm for minmax problem and constraints with quantifiers. In *Proc. of EUROPT21*.

Popov, E.P. (1962). *The Dynamics of Automatic Control Systems*. Pergamon Press.

Qiao, L., Ruan, S., Zhang, G., and Zhang, W. (2018). Robust h2 optimal depth control of an autonomous underwater vehicle with output disturbances and time delay. *Ocean Engineering*.

Ravanbod-Hosseini, L., Noll, D., and Apkarian, P. (2011). Parametric robust h2 control. In *2011 Chinese Control and Decision Conference (CCDC)*, 2056–2061. IEEE.

Scherer, C. and Kose, I. (2006). Robust H_2 estimation with dynamic iqcs: A convex solution. In *Proc. of 45th IEEE Conference on Decision and Control*.

Stoorvogel, A.A. (1993). The robust H_2 control problem: a worst-case design. *IEEE Transactions on Automatic Control*, 1358–1371.

Yang, R., Clement, B., Mansour, A., Li, H.J., and Li, M. (2015). Robust heading control and its application to ciscrea underwater vehicle. In *Proc. of IEEE OCEANS*.